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1989 J. Phys. A: Math. Gen. 22 219

(http://iopscience.iop.org/0305-4470/22/2/011)

J. Phys. A: Math. Gen. 22 (1989) 219-226. Printed in the UK

A family of crossover exponents for the non-linear resistor network

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Received 13 June 1988

Abstract. The crossover exponents $\phi_2(r)$ and $\phi_3(r)$ for the non-linear resistor network are calculated to first order in $\varepsilon = 6 - d$ using the same formalism used by Harris. Our result supports the idea of the analytic continuation proposed by Harris in calculating the non-linear crossover exponent $\phi(r)$ for the non-linear resistor network.

Introduction

Recently, the non-linear resistor network has been investigated by many authors [1-5]. In this model a bond is present with probability p and is absent with probability $1-p$. Each bond is associated with a resistor (of conductance σ_b) which obeys the following equations:

$$[V(\boldsymbol{x}) - V(\boldsymbol{x}')] = \frac{I_{x\to x'}}{\sigma_b^r} |I_{x\to x'}|^{r-1} \tag{1.1}$$

$$\sigma_b[V(\boldsymbol{x}) - V(\boldsymbol{x}')]|V(\boldsymbol{x}) - V(\boldsymbol{x}')|^{s-1} = I_{x\to x'} \tag{1.2}$$

where $V(\boldsymbol{x})$ is the voltage at site $\boldsymbol{x}$, $I_{x\to x'}$ is the current in the bond flowing from site $\boldsymbol{x}$ to site $\boldsymbol{x}'$, r is the non-linear parameter and $s = r^{-1}$.

The conductivity exponent $t(r)$ of the non-linear resistor network is defined as

$$\Sigma(p) \sim |p - p_c|^{t(r)} \tag{1.3}$$

near percolation threshold p_c for a given r, where $\Sigma(p)$ is the bulk conductivity of the sample for the non-linear resistor network. It has been shown [1] using the node-link picture [6, 7] that

$$t(r) = (d - 1 - r^{-1})\nu_p + r^{-1}\phi(r) \tag{1.4}$$

where ν_p is the exponent for the correlation length and $\phi(r)$ is the non-linear crossover exponent governing the scaling behaviour of the two-point resistance $R(\boldsymbol{x}, \boldsymbol{x}')$:

$$[R(\boldsymbol{x}, \boldsymbol{x}')]_{\text{av}} \sim |\boldsymbol{x} - \boldsymbol{x}'|^{\phi(r)/\nu_p} \tag{1.5}$$

where $[\;]_{\text{av}}$ indicates a conditional average, subject to $\boldsymbol{x}$ and $\boldsymbol{x}'$ being in the same cluster.

For the linear resistor networks, there is [8-10] an infinite sequence of crossover exponents $\{\phi_k\}$ needed to completely describe the probability distribution of the two-point resistance $R(\boldsymbol{x}, \boldsymbol{x}')$. Similarly, in the non-linear resistor network, an infinite number of crossover exponents $\{\phi_k(r)\}$ are needed to describe the two-point non-linear resistance $R(\boldsymbol{x}, \boldsymbol{x}')$, where $\phi(r)$ is the first member of $\{\phi_k(r)\}$.

Most recently, Harris [11] has calculated this crossover exponent $\phi(r)$ using the renormalisation group ε-expansion method. As Harris pointed out, although his calculations satisfy several non-trivial self-consistency checks and reproduce known results for $r \to 0$ and $r \to \infty$, they involve an analytic continuation whose status is not beyond question. Accordingly, calculation of the exponent $\phi_2(r)$ associated with w_2 (which will be defined in § 2) would be useful to further test the method of analytic continuation used there. This is the main purpose of this work: to calculate the crossover exponents $\phi_2(r)$ and $\phi_3(r)$ to first order in ε, and to test this method of analytic continuation.

2. Field theory

As discussed in [10] the randomly diluted resistor network can be treated by Stephen's formalism [12]. This formalism has been extended to the non-linear resistor network by Harris [11]. In this paper, we will use a continuum field theory for the model derived by Harris. Since the field theory is quite complex, and has been described in detail in the paper by Harris, we will give a brief review of the derivation. The Hamiltonian of the system is

$$H(\{V\}) = \sum_{\langle x,x'\rangle} \frac{1}{s+1} \sigma_b |V(x) - V(x')|^{s+1} \tag{2.1}$$

where the summation is over the nearest-neighbour sites and b is the bond connecting site x and x'.

The replicated effective Hamiltonian H_{eff} is defined as

$$\exp(-H_{\text{eff}}) = \left[\prod_{\alpha=1}^{n} \exp[-H(\{V_\alpha\})]\right]_{\text{av}} \tag{2.2}$$

or

$$H_{\text{eff}} = -\ln\left[\prod_{\alpha=1}^{n} \exp(-H)\right]_{\text{av}} \tag{2.3}$$

where $[\ ]_{\text{av}}$ denotes the average over the random configurations, $\alpha = 1, 2, \ldots, n$ labels replicas and we have introduced n replicas to facilitate the random average.

Now we consider the correlation function $G(x, x', \lambda)$ which can be defined as [11]

$$G(x, x', \lambda) \equiv \int DV \exp[-H(\{V\})] \exp\{i\lambda[V(x) - V(x')]\} \tag{2.4}$$

where DV indicates an integration over all variables $\{V(x)\}$.

In order that the dominant contribution of (2.4) is determined by (1.1), we should continue λ into large imaginary values, i.e. $\lambda = i\lambda_0$ with $\lambda_0 \to \infty$. So the correlation function becomes

$$G(x, x', \lambda) = \left[\exp\left(\prod_{\alpha=1}^{n} \lambda_{\alpha,0}^{r+1} R(x, x')/(r+1)\right)\right]_{\text{av}} \qquad \lambda_{\alpha,0} \to \infty \tag{2.5}$$

where $\lambda_\alpha = i\lambda_{\alpha,0}$. When λ_0 is near the positive real axis and

$$|\lambda_0^{r+1}|/\sigma_0^r \gg 1 \tag{2.6a}$$

$$n|\lambda_0^{r+1}|/\sigma_0^r \ll 1 \tag{2.6b}$$

we have

$$G(x, x', \lambda) = \left[\nu(x, x')\left(1 + R(x, x') \sum_{\alpha} (-\mathrm{i}\lambda_\alpha)^{r+1}/(r+1)\right)\right]_{\mathrm{av}} \tag{2.7}$$

where $\nu(x, x')$ is one if x and x' are in the same cluster and zero otherwise. Here we have assumed that all the present bonds have the same conductance σ_0 in bond dilution. Therefore, the non-linear resistor network can be formulated as a crossover from the percolation problem as for the linear case [10, 12].

To get a field theory we transform (2.3) into its Fourier components. We find that

$$H_{\mathrm{eff}} = -\sum_{\langle x,x'\rangle} \sum_{\lambda} B_{\lambda} \psi_{\lambda}(x) \psi_{-\lambda}(x') \tag{2.8}$$

where $\boldsymbol{\lambda}$ is the Fourier parameter, $\psi_{\lambda}(x)$ is the order parameter defined by

$$\psi_{\lambda}(x) = \exp[\mathrm{i}\boldsymbol{\lambda} \cdot \boldsymbol{V}(x)] \tag{2.9}$$

and

$$B_{\lambda} \sim \frac{1}{z[r(\boldsymbol{\lambda}) - 1]} \tag{2.10}$$

where z is the coordination number of the lattice. Near criticality

$$\begin{aligned} r(\boldsymbol{\lambda}) &= r(0) - \sum_{k=1}^{\infty} w_k \left(\sum_{\alpha} (-\mathrm{i}\lambda_\alpha)^{r+1} \right)^k \\ &\equiv r(0) + \delta r(\boldsymbol{\lambda}) \end{aligned} \tag{2.11}$$

where the w_k are constants, $w_k \sim (\sigma_0^r)^{-(2k-1)}$ and $r(0) \sim p - p_c$. The exponent $\phi_k(r)$ is the crossover exponent associated with w_k in (2.11).

3. ε expansion

In this section we will use the momentum-shell renormalisation group recursion relation [13] to calculate the non-linear crossover exponent for the non-linear resistor network. The recursion relation for $r(\boldsymbol{\lambda})$ can be obtained by integrating out degrees of freedom with wavenumber in the annulus $b^{-1}\Lambda < q < \Lambda = 1$, where Λ is a cutoff determined by the lattice constant a such that $a\Lambda \sim 1$, and rescaling the field via $\psi(\boldsymbol{q}/b) \to b^{(d-2+\eta)/2}\psi(\boldsymbol{q})$, where $\psi(\boldsymbol{q})$ is the order parameter field in Fourier space. Eliminating an infinitesimal shell at each iteration with $b = \mathrm{e}^{\delta l}$, we obtain the recursion relation as [11]

$$\frac{\mathrm{d}r(\boldsymbol{\lambda})}{\mathrm{d}l} = (2 - \eta_p) r(\boldsymbol{\lambda}) - g\Sigma(\boldsymbol{\lambda}) \tag{3.1a}$$

where [14]

$$\eta_p = -\varepsilon/21 \qquad g = 2\varepsilon/7 \qquad \nu_p = \tfrac{1}{2} + 5\varepsilon/84 \tag{3.1b}$$

and $\Sigma(\boldsymbol{\lambda})$ is given by

$$\Sigma(\boldsymbol{\lambda}) \equiv -2G(\boldsymbol{\lambda})G(0) + \tilde{\Sigma}(\boldsymbol{\lambda}) \tag{3.2}$$

where $G(\boldsymbol{\lambda})$ is the mean field propagator evaluated at $q^2 = 1$:

$$G(\boldsymbol{\lambda})^{-1} = 1 + r(0) + \delta r(\boldsymbol{\lambda}) \tag{3.3}$$

and

$$\tilde{\Sigma}(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\tau}} G(\boldsymbol{\lambda} - \boldsymbol{\tau}) G(\boldsymbol{\tau}) \tag{3.4}$$

$$= \sum_{\boldsymbol{\tau}} G(\tfrac{1}{2}\boldsymbol{\lambda} - \boldsymbol{\tau}) G(\tfrac{1}{2}\boldsymbol{\lambda} + \boldsymbol{\tau}). \tag{3.5}$$

For the non-linear resistor network, we will consider the analytic continuation of the recursion relation (3.1a) for $\boldsymbol{\lambda}$ in the regime described by (2.6). In order to calculate $\phi_k(r)$, we expand (3.5) in powers of w_k:

$$\tilde{\Sigma}(\boldsymbol{\lambda}) \sim \sum_{\boldsymbol{\tau}} \left[G_0(\tfrac{1}{2}\boldsymbol{\lambda} + \boldsymbol{\tau}) + w_k G_0^2(\tfrac{1}{2}\boldsymbol{\lambda} + \boldsymbol{\tau}) \left(\sum_{\alpha} (-\tfrac{1}{2}\mathrm{i}\lambda_\alpha + \mathrm{i}\tau_\alpha)^{r+1} \right)^k \right]$$
$$\times \left[G_0(\tfrac{1}{2}\boldsymbol{\lambda} - \boldsymbol{\tau}) + w_k G_0^2(\tfrac{1}{2}\boldsymbol{\lambda} - \boldsymbol{\tau}) \left(\sum_{\alpha} (-\tfrac{1}{2}\mathrm{i}\lambda_\alpha - \mathrm{i}\tau_\alpha)^{r+1} \right)^k \right]$$
$$= \sum_{\boldsymbol{\tau}} G_0(\tfrac{1}{2}\boldsymbol{\lambda} + \boldsymbol{\tau}) G_0(\tfrac{1}{2}\boldsymbol{\lambda} - \boldsymbol{\tau}) + 2 w_k \sum_{\boldsymbol{\tau}} \left(\sum_{\alpha} (-\tfrac{1}{2}\mathrm{i}\lambda_\alpha + \mathrm{i}\tau_\alpha)^{r+1} \right)^k$$
$$\times G_0^2(\tfrac{1}{2}\boldsymbol{\lambda} + \boldsymbol{\tau}) G_0(\tfrac{1}{2}\boldsymbol{\lambda} - \boldsymbol{\tau}) \tag{3.6}$$

where

$$G_0^{-1}(\boldsymbol{\lambda}) = 1 - w_1 \sum_{\alpha} (-\lambda_\alpha)^{r+1}. \tag{3.7}$$

Note that

$$G_0^k(\boldsymbol{\lambda}) = \frac{1}{(k-1)!} \int_0^\infty u^{k-1}\, \mathrm{d}u\, \exp[-u G_0^{-1}(\boldsymbol{\lambda})]$$
$$= \frac{1}{(k-1)!} \int_0^\infty u^{k-1}\, \mathrm{d}u\, \exp\left[-u \left(1 - w_1 \sum_{\alpha} (-\lambda_\alpha)^{r+1} \right) \right].$$

So we obtain

$$\tilde{\Sigma}_k = 2 w_k \int_0^\infty u\, \mathrm{d}u\, \mathrm{e}^{-u} \int_0^\infty \mathrm{d}\nu\, \mathrm{e}^{-\nu} \sum_{\boldsymbol{\tau}} \exp\left[w_1 \sum_{\alpha} \left(u(-\tfrac{1}{2}\mathrm{i}\lambda_\alpha + \mathrm{i}\tau_\alpha)^{r+1} \right.\right.$$
$$\left.\left. + \nu(-\tfrac{1}{2}\mathrm{i}\lambda_\alpha - \mathrm{i}\tau_\alpha)^{r+1} \right) \right] \left(\sum_{\alpha} (-\tfrac{1}{2}\mathrm{i}\lambda_\alpha + \mathrm{i}\tau_\alpha)^{r+1} \right)^k \tag{3.8}$$

where we have dropped $\boldsymbol{\lambda}$ in $\tilde{\Sigma}$ for notational convenience. Now we change the variable from τ_α to μ_α, i.e.

$$\tau_\alpha = \mu_\alpha + \tfrac{1}{2}\lambda_\alpha \frac{u^s - \nu^s}{u^s + \nu^s} \tag{3.9}$$

where $s = r^{-1}$, so that

$$\tilde{\Sigma}_k = 2 w_k \int_0^\infty u\, \mathrm{d}u\, \mathrm{e}^{-u} \int_0^\infty \mathrm{d}\nu\, \mathrm{e}^{-\nu} \sum_{\boldsymbol{\mu}} \mathrm{e}^A B_k \tag{3.10}$$

where

$$A \equiv w_1 \sum_{\alpha} \left[u \left(\frac{-\mathrm{i}\lambda_\alpha}{u^s + \nu^s} \nu^s + \mathrm{i}\mu_\alpha \right)^{r+1} + \nu \left(\frac{-\mathrm{i}\lambda_\alpha}{u^s + \nu^s} u^s - \mathrm{i}\mu_\alpha \right)^{r+1} \right] \tag{3.11}$$

and

$$B_k \equiv \left[\sum_\alpha \left(\frac{-\mathrm{i}\lambda_\alpha}{u^s+\nu^s}\nu^s + \mathrm{i}\mu_\alpha\right)^{r+1}\right]^k. \tag{3.12}$$

Since λ_α is large (or $w_1\lambda_0^{1+r}$ is large) we can expand A in powers of $1/\lambda_\alpha$:

$$\begin{aligned} A &= w_1 \sum_\alpha (-\mathrm{i}\lambda_\alpha)^{r+1} \frac{u\nu}{(u^s+\nu^s)^{r+1}}\left[\nu^s\left(1+\frac{\mathrm{i}\mu_\alpha(u^s+\nu^s)}{-\mathrm{i}\lambda_\alpha \nu^s}\right)^{r+1} + u^s\left(1-\frac{\mathrm{i}\mu_\alpha(u^s+\nu^s)}{-\mathrm{i}\lambda_\alpha u^s}\right)^{r+1}\right] \\ &= w_1\sum_\alpha(-\mathrm{i}\lambda_\alpha)^{r+1}\frac{u\nu}{(u^s+\nu^s)^r} - w_1\sum_\alpha(-\mathrm{i}\lambda_\alpha)^{r-1}\mu_\alpha^2 F_0 \end{aligned} \tag{3.13}$$

where

$$F_0 \equiv \tfrac{1}{2}r(r+1)(u^s+\nu^s)^{2-r}\frac{u\nu}{u^s\nu^s} \tag{3.14}$$

so that e^A has a Gaussian form. Similarly, we have

$$\begin{aligned} B_k = \Bigg\{\sum_\alpha\left(\frac{-\mathrm{i}\lambda_\alpha}{u^s+\nu^s}\nu^s\right)^{r+1}\Bigg[1 &+ (r+1)\frac{\mathrm{i}\mu_\alpha}{-\mathrm{i}\lambda_\alpha\nu^s}(u^s+\nu^s) \\ &+ \frac{r(r+1)}{2}\left(\frac{\mathrm{i}\mu_\alpha}{-\mathrm{i}\lambda_\alpha\nu^s}(u^s+\nu^s)\right)^2 + \ldots\Bigg]\Bigg\}^k. \end{aligned} \tag{3.15}$$

3.1. Calculation for $\phi_2(r)$

The procedure for calculating $\phi_k(r)$ is very similar to that for obtaining the linear exponent ϕ_k. Setting $k=2$, we expand (3.15) and keep only even powers of μ_α because of the Gaussian form of e^A. It is easy to show that μ^4 or higher-order terms will not contribute to $\phi_2(r)$ in the scaling region. We thus have

$$B_2 = B_2^{(1)} + B_2^{(2)} + B_2^{(3)}$$

where

$$B_2^{(1)} \equiv \left(\sum_\alpha(-\mathrm{i}\lambda_\alpha)^{r+1}\right)^2 \frac{\nu^{2s+2}}{(u^s+\nu^s)^{2r+2}} \tag{3.16}$$

$$B_2^{(2)} \equiv -\left(\sum_\alpha(-\mathrm{i}\lambda_\alpha)^r\mu_\alpha\right)^2\frac{(r+1)^2\nu^2}{(u^s+\nu^s)^{2r}} \tag{3.17}$$

and

$$B_2^{(3)} \equiv -\left(\sum_\alpha(-\mathrm{i}\lambda_\alpha)^{r+1}\right)\left(\sum_\beta(-\mathrm{i}\lambda_\beta)^{r-1}\mu_\beta^2\right)\frac{r(r+1)\nu^2}{(u^s+\nu^s)^{2r}}. \tag{3.18}$$

Substituting (3.14) and (3.15) into (3.10) we obtain

$$\begin{aligned} \tilde{\Sigma}_2 = 2w_2\int_0^\infty u\,\mathrm{d}u\,\mathrm{e}^{-u}\int_0^\infty \mathrm{d}\nu\,\mathrm{e}^{-\nu}\exp\left(w_1\sum_\alpha(-i\lambda_\alpha)^{r+1}\frac{u\nu}{(u^s+\nu^s)^r}\right) \\ \times\int D\mu_\alpha \exp\left(-w_1F_0\sum_\alpha(-\mathrm{i}\lambda_\alpha)^{r-1}\mu_\alpha^2\right)(B_2^{(1)}+B_2^{(2)}+B_2^{(3)}). \end{aligned} \tag{3.19}$$

Before calculating $\phi_2(r)$, we make the following expansion in (3.19):

$$\exp\left(w_1\sum_\alpha(-\mathrm{i}\lambda_\alpha)^{r+1}\frac{u\nu}{(u^s+\nu^s)^r}\right) = 1 + w_1\sum_\alpha(-\mathrm{i}\lambda_\alpha)^{r+1}\frac{u\nu}{(u^s+\nu^s)^r}+\ldots. \tag{3.20}$$

We will keep the correct term to get the scaling form $w_2[\Sigma_\alpha(-i\lambda_\alpha)^{r+1}]^2$ in the calculation below. We first calculate the contribution from $B_2^{(1)}$. Note that $B_2^{(1)}$ does not depend on μ_α and $\int D\mu_\alpha \exp[-a\Sigma_\alpha(-i\lambda_\alpha)^{r-1}\mu_\alpha^2] \sim 1 + nO(1)$, where a is a constant. Keeping the first term in (3.20), we obtain

$$\tilde{\Sigma}_2^{(1)} = 2w_2\left(\sum_\alpha(-i\lambda_\alpha)^{r+1}\right)^2 \int_0^\infty u\,du\,e^{-u}\int_0^\infty d\nu\,e^{-\nu}\frac{\nu^{2s+2}}{(u^s+\nu^s)^{2r+2}}$$

$$= w_2\left(\sum_\alpha(-i\lambda_\alpha)^{r+1}\right)^2 C^{(1)}(r) \tag{3.21}$$

where

$$C^{(1)}(r) \equiv \int_{-1}^{1} dy\,\frac{(1+y)^{2+2/r}(1-y)}{[(1+y)^{1/r}+(1-y)^{1/r}]^{2r+2}}. \tag{3.22}$$

The contribution from $B_2^{(2)}$ can be calculated as follows.

From (3.17) and (3.19), we have

$$\tilde{\Sigma}_2^{(2)} = -2w_2\int_0^\infty u\,du\,e^{-u}\int_0^\infty d\nu\,e^{-\nu}\exp\left(w_1\sum_\alpha(-i\lambda_\alpha)^{r+1}\frac{u\nu}{(u^s+\nu^s)^r}\right)$$
$$\times(r+1)^2\frac{\nu^2}{(u^s+\nu^s)^{2r}}\left[\int D\mu_\alpha\exp\left(-w_1F_0\sum_\alpha(-i\lambda_\alpha)^{r-1}\mu_\alpha^2\right)\right.$$
$$\left.\times\sum_\alpha(-i\lambda_\alpha)^{2r}\mu_\alpha^2\right] \tag{3.23}$$

since

$$\int D\mu_\alpha\exp\left(-w_1F_0\sum_\alpha(-i\lambda_\alpha)^{r-1}\mu_\alpha^2\right)\mu_\beta^2 = \frac{1}{2w_1F_0(-i\lambda_\beta)^{r-1}} \qquad n\to 0. \tag{3.24}$$

We substitute (3.24) into (3.23) and, in order to cancel w_1 in (3.24), we keep the second term in the expansion (3.20). Therefore we may write $\tilde{\Sigma}_2^{(2)}$ as

$$\tilde{\Sigma}_2^{(2)} = -w_2\left(\sum_\alpha(-i\lambda_\alpha)^{r+1}\right)^2\int_0^\infty u\,du\,e^{-u}\int_0^\infty d\nu\,e^{-\nu}\frac{2(r+1)}{r}\frac{u^s\nu^{s+2}}{(u^s+\nu^s)^{2r+2}}$$

$$= w_2\left(\sum_\alpha(-i\lambda_\alpha)^{r+1}\right)^2 C^{(2)}(r) \tag{3.25}$$

where

$$C^{(2)}(r) \equiv -\int_{-1}^{1} dy\,\frac{(r+1)}{r}\frac{(1-y^2)^{1/r+1}(1+y)}{[(1+y)^{1/r}+(1-y)^{1/r}]^{2r+2}}. \tag{3.26}$$

The contribution from $B_2^{(3)}$ can be calculated as follows.

From (3.24) one can see that

$$\sum_\beta\int D\mu_\alpha\exp\left(-w_1F_0\sum_\alpha(-i\lambda_\alpha)^{r-1}\mu_\alpha^2\right)(-i\lambda_\beta)^{r-1}\mu_\beta^2 = \sum_\beta\frac{1}{2w_1F_0} = 0 \qquad n\to 0.$$

Hence $B_2^{(3)}$ does not contribute to $\phi_2(r)$.

From (2.11), (3.1), (3.21) and (3.26) we have

$$\frac{dw_2}{dl} = (2-\eta)w_2 - \tfrac{1}{2}g[2+C_2(r)]w_2 \equiv \frac{\phi_2(r)}{\nu_p}w_2$$

or

$$\phi_2(r) = 1 + \tfrac{1}{14}\varepsilon C_2(r) \tag{3.27}$$

where

$$C_2(r) = \int_{-1}^{1} \mathrm{d}y \left((1+y)^{1/r} - \frac{r+1}{r}(1-y)^{1/r} \right) \frac{(1+y)^{2+1/r}(1-y)}{[(1+y)^{1/r} + (1-y)^{1/r}]^{2r+2}}. \tag{3.28}$$

The calculation of $\phi_3(r)$ is similar to that of $\phi_2(r)$. Here we should expand B_3 to fourth order in μ_α, and keep the proper term in (3.20) to get the correct scaling form $w_3[\Sigma_\alpha (-i\lambda_\alpha)^{r+1}]^3$. We obtain

$$\phi_3(r) = 1 + \tfrac{1}{14}\varepsilon C_3(r) \tag{3.29}$$

where

$$C_3(r) = \int_{-1}^{1} \mathrm{d}y \frac{(1+y)^{3+1/r}(1-y)}{[(1+y)^{1/r} + (1-y)^{1/r}]^{3r+3}} \times \left((1+y)^{2/r} - \frac{3(r+1)}{r}(1+y)^{1/r}(1-y)^{1/r} + \frac{3(r+1)}{2r}(1-y)^{2/r} \right). \tag{3.30}$$

In principle, $\phi_k(r)$ for $k > 3$ can also be calculated in the same way. Now we check several limits of r.

(i) $r \to 1$: we obtain $C_2(1) = 0$ and $C_3(1) = -\frac{1}{35}$ which agrees with ϕ_k for the linear resistor network [10].

(ii) $r \to \infty$: we have $\phi_2(\infty) = \phi_3(\infty) = 1$, which is expected [2, 15].

Finally, we note that for the limit $r \to 0$, which corresponds to the exponent of the chemical length [2, 11], we obtain $C_2(0) = C_3(0) = \frac{1}{2}$, which provides strong evidence that there is not a hierarchy of exponents for the chemical length.

In summary, we have calculated the non-linear crossover exponents $\phi_2(r)$ and $\phi_3(r)$ which provide a useful test for the method of analytic continuation proposed by Harris in calculating the non-linear crossover exponent $\phi(r)$ for the non-linear resistor network.

Acknowledgments

I would like to thank Professor A B Harris for helpful discussions. I also thank the referees for useful comments. I thank the National Science Foundation, MRL programme, for support under grant no DMR 85-19059.

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